SYMMETRIC CURVES, HEXAGONS, AND THE GIRTH OF SPHERES IN DIMENSION 3

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ABSTRACT

It is proved that every centrally symmetric simple closed curve on the boundary of a centrally symmetric convex body in a three-dimensional linear space possesses an inscribed concentric affinely regular hexagon. This result is used to settle affirmatively a conjecture in [2] about the metric structure of the unit spheres of three-dimensional normed space.

The preceding abstract will do for an introduction. The idea of applying a topological device such as is used in Theorem 1 to the proof of the conjecture was inspired by a similar attempt by L. Danzer (unpublished notes), who had arrived at the problem independently. The author is indebted to M. Sebastiani for help with the algebraic topology in the proof of Theorem 1.

1. **Inscribed hexagons.** Let E be a real linear space with $\dim E = 3$, provided with the natural Hausdorff topology. In this paper, a *simple closed curve* is the homeomorphic image of a circle; a *parametrized curve* in a subset F of E is a continuous function $f:[\alpha,\beta]\to F$.

1. THEOREM. *Let K be a convex body (compact convex set with non-empty interior) in E, with $-K=K$, and let ∂K be its boudary. If $C\subset\partial K$ is a simple closed curve such that $-C=C$, then C contains the vertices of an affinely regular hexagon centred at* 0.

Proof. 1. It is sufficient to show that there exists $p\in C$ such that $C\cap(C-p)\neq\emptyset$: for if q is in the non-empty intersection, p, $p+q$, q, $-p$, $-p-q$, $-q$ are the required vertices. We shall therefore assume

$$C\cap(C-p)=\emptyset,\qquad p\in C \tag{1}$$

and derive a contradiction.

2. By the Jordan Curve Theorem, $\partial K\backslash C$ consists of two components; the mapping $u\to -u:\partial K\to\partial K$ maps C onto itself; since it is orientation-preserving on C and orientation-reversing on ∂K, it interchanges the components of $\partial K\backslash C$, which may therefore be called A and $-A$.

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For every parametrized curve $f:[0,1] \to E\backslash C$ with endpoints $f(0), f(1) \in E\backslash \partial K$ the intersection numbers modulo 2 $i(f,A)$, $i(f,-A)$, $i(f,\partial K)$ are well defined and constant under homotopies of such curves; they satisfy

$$(2) \qquad i(f,A) + i(f,-A) = i(f,\partial K);$$

the second member is 0 or 1 according as the endpoints of f are in the same component or different components of $E\backslash \partial K$.

Indeed, consider the commutative diagram

$$\begin{array}{ccccc} H_1(E\backslash C, E\backslash \partial K) & \leftrightarrow H_c^2(A \cup -A) = & H_c^2(A) \oplus H_c^2(-A) & \leftrightarrow & Z_2 \oplus Z_2 \\ \downarrow & & \downarrow & & \downarrow \\ H_1(E, E\backslash \partial K) & \leftrightarrow & H^2(\partial K) & \leftrightarrow & Z_2 \end{array}$$

where the coefficients are in Z_2, the subscript c denotes compact support, the left-hand horizontal arrows indicate the Alexander-Pontrjagin duality [1], and all the mappings are natural. The image of the homology class of a parametrized curve as above under the upper [lower] row of isomorphisms is, by definition, $i(f,A) \oplus i(f,-A)$ [$i(f,\partial K)$]. The preceding statements follow. Intersection numbers modulo 2 are used to avoid questions of orientation.

3. There exists a parametrized curve $h:[0,2] \to \partial K$ such that $h([0,2]) = C$, h is injective except for $h(2) = h(0)$, and $h(t+1) = -h(t)$, $0 \leqq t \leqq 1$. For each u, $0 \leqq u \leqq 1$, we define

$$f_u(t) = h(t+u) - h(u), \qquad 0 \leqq t \leqq 1.$$

Now (1) implies that $f_u(t) \notin C$ for all u, t, $0 \leqq u$, $t \leqq 1$. Therefore f_u is a parametrized curve in $E\backslash C$ for every u; its endpoints are $f_u(0) = 0$ and $f_u(1) = h(1+u) - h(u) = -2h(u) \in 2\partial K$, so that they belong to different components of $E\backslash \partial K$; and the mapping $u \to f_u$ is a homotopy.

Consequently, using part 2 of the proof, $u \to i(f_u, A)$ is constant, and therefore

$$(3) \qquad i(f_1, A) = i(f_0, A);$$

by (2) and the location of the endpoints,

$$(4) \qquad i(f_0, A) + i(f_0, -A) = i(f_0, \partial K) = 1.$$

Finally, $f_1(t) = h(t+1) - h(1) = -h(t) = -h(t) + h(0) = f_0(t)$, $0 \leqq t \leqq 1$; central symmetry then implies

$$(5) \qquad i(f_1, A) = i(f_0, -A).$$

Using (3), (5), (4) in succession, we obtain the contradiction

$$0 = i(f_0,A) + i(f_0,A) = i(f_0,A) + i(f_1,A) = i(f_0,A) + i(f_0,-A) = 1.$$

A careful perusal of the proof shows that the assumption that K is a symmetric convex body may be replaced by the weaker assumption that K is a symmetric compact set that is star-shaped at the interior point 0. It is then possible to state Theorem 1 thus generalized in a way that makes no explicit mention of K. For this purpose, call a set $F \subset E$ *simple at* 0 if every ray issuing from 0 meets F in at most one point.

2. THEOREM. *If $C \subset E$ is a simple closed curve that satisfies $-C = C$ and is simple at* 0, *it contains the vertices of an affinely regular hexagon centred at* 0.

Proof. There exists a symmetric compact K, star-shaped at the interior point 0, such that $C \subset \partial K$. The conclusion then follows from the generalized form of Theorem 1 discussed above.

We sketch a construction for K. Let $\| \;\; \|$ be a norm in E, and let Σ be the corresponding unit ball. The mapping $\mathrm{sgn}: x \to \|x\|^{-1}x$: $E \backslash \{0\} \to \partial\Sigma$, when restricted to the compact set C, is continuous and injective (since C is simple at 0), and hence reduces there to a homeomorphism σ of C onto $\mathrm{sgn}(C)$, a simple closed curve in $\partial\Sigma$. Clearly, $-\mathrm{sgn}(C) = \mathrm{sgn}(C)$ and $\sigma(-x) = -\sigma(x)$, $x \in C$. As in the proof of Theorem 1, $\partial\Sigma \backslash \mathrm{sgn}(C)$ consists of two components, say Γ and $-\Gamma$; the Jordan Curve Theorem further implies that there exist homeomorphisms of $\Gamma \cup \mathrm{sgn}(C)$ and of $-\Gamma \cup \mathrm{sgn}(C)$ onto a euclidean plane disk that map $\mathrm{sgn}(C)$ onto the boundary.

For each $p \in \mathrm{sgn}(C)$ we set $\phi(p) = \|\sigma^{-1}(p)\|$, so that

$$\phi(-p) = \phi(p) > 0 \tag{6}$$

for all these points. On account of the homeomorphisms just mentioned, ϕ can be extended to a continuous positive-valued function defined on all $\partial\Sigma$; we may assume that it satisfies (6) everywhere, since otherwise we should replace $\phi(p)$ by $\frac{1}{2}(\phi(p) + \phi(-p))$, thus leaving the function unchanged on $\mathrm{sgn}(C)$.

It may then be verified directly that the star-shaped set $K = \{0\} \cup \{x \in E \backslash \{0\}: \|x\| \leqq \phi(\mathrm{sgn}\, x)\}$ satisfies all the requirements.

When C is contained in a plane, each point of C is obviously a vertex of some inscribed affinely regular hexagon. The referee suggests the query: Are Theorems 1 and 2 best possible in this sense in the non-planar case?

2. **The girth of spheres.** Let X be a finite-dimensional normed real linear space with norm $\| \;\; \|$, and let Σ be its unit ball, with the boundary $\partial\Sigma$. In [2] we defined the *girth* of Σ to be $2m(X)$, where $m(X) = \min\{\delta(-p, p): p \in \partial\Sigma\}$ and δ denotes the inner metric of $\partial\Sigma$ induced by the norm. In [2, Lemma 5.1] it is shown that the minimum is attained, and that

(7) $2m(X) = \min\{l(C): C$ a rectifiable simple closed curve in $\partial\Sigma$, with $-C = C\}$, where $l(C)$ denotes the length of C.

3. THEOREM. *If* $\dim X = 3$, *then* $m(X) \geqq 3$.

Proof. Let C be a rectifiable simple closed curve in $\partial\Sigma$ with $-C = C$. By Theorem 1, C contains the vertices of an affinely regular hexagon centred at 0;

let these be $p_1, p_2, p_3, p_4, p_5, p_6$, in the order in which they appear along C for some given orientation and starting point.

Now vertices that are consecutive in this ordering need not be adjacent in the hexagon; however, $\| p_i \| = 1,\ i = 1,\cdots,6$; and for any distinct $i,j = 1,\cdots,6$, either $p_i,\ p_j$ are adjacent and $\| p_j - p_i \| = 1$, or $-p_i,\ p_j$ are adjacent and $\| p_j - p_i \| \geqq \| 2p_i \| - \| p_j - (-p_i) \| \geqq 2 - 1 = 1$, or p_i, p_j are opposite and $\| p_j - p_i \| = 2$, so that $\| p_j - p_i \| \geqq 1$ in any case.

Setting $p_0 = p_6$, we find $l(C) \geqq \sum_1^6 \| p_j - p_{j-1} \| \geqq 6$, and the conclusion follows from (7).

In [2] we further defined $M(X) = \max\{\delta(-p,p): p \in \partial\Sigma\}$, $D(X) = \max\{\delta(p,q): p,q \in \partial\Sigma\}$, as well as for $n = 2,3,\cdots$,

$$m_*(n) = \min\{m(X): \dim X = n\},\ M_*(n) = \min\{M(X): \dim X = n\},$$

$$D_*(n) = \min\{D(X): \dim X = n\}$$

and the corresponding maxima; all are attained, by [2, Theorem 8.2].

4. THEOREM. $m_*(3) = M_*(3) = D_*(3) = 3.$

Proof. By Theorem 3, $3 \leqq m_*(3)$; trivially, $m_*(3) \leqq M_*(3)$; and [2, Theorem 8.5, (b)] gives $M_*(3) = D_*(3) \leqq 3$.

The equality $m_*(3) = 3$ was the purport of [2, Conjecture 9.4, (b)]; cf. also [3, p. 82].

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